EXPONENTIAL MAP AND L_∞ ALGEBRA ASSOCIATED TO A LIE PAIR

CAMILLE LAURENT-GENGOUX, MATHIEU STIÉNON, AND PING XU

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Abstract. In this note, we unveil homotopy-rich algebraic structures generated by the Atiyah classes relative to a Lie pair (L,A) of algebroids. In particular, we prove that the quotient L/A of such a pair admits an essentially canonical homotopy module structure over the Lie algebroid A, which we call Kapranov module.

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1. Kapranov modules

Let A be a Lie algebroid (either real or complex) over a manifold M with anchor ρ. By an A-module, we mean a module of the corresponding Lie-Rinehart algebra $\Gamma(A)$ over the associative algebra $C^\infty(M)$. An A-connection on a smooth vector bundle E over M is a bilinear map $\nabla:\Gamma(A)\otimes\Gamma(E)\to\Gamma(E)$ satisfying $\nabla_{fa}e=f\nabla_a e$ and $\nabla_a(fe)=\big(\rho(a)f\big)e+f\nabla_a e$, for all $a\in\Gamma(A)$, $e\in\Gamma(E)$, and $f\in C^\infty(M)$. A vector bundle E endowed with a *flat* A-connection (also known as an infinitesimal A-action) is an A-module; more precisely, its space of smooth sections $\Gamma(E)$ is one.
Atiyah class. Given a Lie pair (L,A) of algebroids, *i.e.* a Lie algebroid L with a Lie subalgebroid A, the Atiyah class α_E of an A-module E relative to the pair (L,A) is defined as the obstruction to the existence of an A-*compatible* L-connection on E. An L-connection ∇ is A-compatible if its restriction to $\Gamma(A)\otimes\Gamma(E)$ is the given infinitesimal A-action on E and $\nabla_a\nabla_l-\nabla_l\nabla_a=\nabla_{[a,l]}$ for all $a\in\Gamma(A)$ and $l\in\Gamma(L)$. This fairly recently defined class (see [3]) has as double origin, which it generalizes, the Atiyah class of holomorphic vector bundles and the Molino class of foliations. The quotient L/A of the Lie pair (L,A) is an A-module [3]. Its Atiyah class $\alpha_{L/A}$ can be described as follows. Choose an L-connection ∇ on L/A extending the A-action. Its curvature is the vector bundle map $R^\nabla:\wedge^2L\to\operatorname{End}(E)$ defined by $R^\nabla(l_1,l_2)=\nabla_{l_1}\nabla_{l_2}-\nabla_{l_2}\nabla_{l_1}-\nabla_{[l_1,l_2]}$, for all $l_1,l_2\in\Gamma(L)$. Since L/A is an A-module, R^∇ vanishes on $\wedge^2A$ and, therefore, determines a section $R^\nabla_{L/A}$ of $A^*\otimes(L/A)^*\otimes\operatorname{End}(L/A)$. It was proved in [3] that $R^\nabla_{L/A}$ is a 1-cocycle for the Lie algebroid A with values in the A-module $(L/A)^*\otimes\operatorname{End}(L/A)$ and that its cohomology class $\alpha_{L/A}\in H^1\big(A;(L/A)^*\otimes\operatorname{End}(L/A)\big)$ is independent of the choice of the connection.

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Kapranov modules over a Lie algebroid. Let M be a smooth manifold, and let R be the algebra of smooth functions on M valued in $\mathbb{R}$ (or $\mathbb{C}$). Let A be a Lie algebroid over M. The Chevalley-Eilenberg differential d_A and the exterior product make $\Gamma(\wedge^\bullet A^*)$ into a differential graded commutative R-algebra.

Now let E be a smooth vector bundle over M. Deconcatenation defines an R-coalgebra structure on $\Gamma(S^\bullet E)$. The comultiplication $\Delta : \Gamma(S^\bullet E) \to \Gamma(S^\bullet E) \otimes_R \Gamma(S^\bullet E)$ is given by

$$\Delta(e_1 \odot e_2 \odot \cdots \odot e_n) = \sum_{p+q=n} \sum_{\sigma \in \mathfrak{S}_p^q} (e_{\sigma(1)} \odot \cdots \odot e_{\sigma(p)}) \otimes (e_{\sigma(p+1)} \odot \cdots \odot e_{\sigma(n)}),$$

for any $e_1, \dots, e_n \in \Gamma(E)$. Let $\mathfrak{e}$ denote the ideal of $\Gamma(S^\bullet(E^*))$ generated by $\Gamma(E^*)$. The algebra $\operatorname{Hom}_R\big(\Gamma(S^\bullet E), R\big)$ dual to the coalgebra $\Gamma(S^\bullet E)$ is the $\mathfrak{e}$-adic completion of $\Gamma(S^\bullet(E^*))$. It will be denoted by $\Gamma(\hat{S}^\bullet(E^*))$. Equivalently, one can think of the completion $\hat{S}^\bullet(E^*)$ of $S^\bullet(E^*)$ as a bundle of algebras over M. Note that $\Gamma(\wedge^\bullet A^*)$ is an R-subalgebra of $\Gamma(\wedge^\bullet A^* \otimes \hat{S}^\bullet E^*)$.

Recall that an $L_\infty[1]$ algebra is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ endowed with a sequence $(\lambda_k)_{k=1}^\infty$ of symmetric multilinear maps $\lambda_k : \otimes^k V \to V$ of degree 1 satisfying the generalized Jacobi identity

$$\textstyle\sum_{k=1}^n \sum_{\sigma\in\mathfrak{S}_k^{n-k}} \varepsilon(\sigma; v_1, \cdots, v_n)\, \lambda_{1+n-k}\big(\lambda_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)}), v_{\sigma(k+1)}, \cdots, v_{\sigma(n)}\big) = 0$$

for each $n \in \mathbb{N}$ and for any homogeneous vectors $v_1, v_2, \dots, v_n \in V$. Here $\mathfrak{S}_p^q$ denotes the set of (p,q)-*shuffles*[1] and $\varepsilon(\sigma; v_1, \cdots, v_n)$ the *Koszul sign*[2] of the permutation σ of the (homogeneous) vectors $v_1, v_2, \dots, v_n$.

Definition 1.1. *A Kapranov module over a Lie algebroid $A \to M$ is a vector bundle $E \to M$ together with an $L_\infty[1]$ algebra structure on $\Gamma(\wedge^\bullet A^* \otimes E)$ defined by a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of multibrackets (called Kapranov multibrackets) such that (1) the unary bracket $\lambda_1 : \Gamma(\wedge^\bullet A^* \otimes E) \to \Gamma(\wedge^{\bullet+1} A^* \otimes E)$ is the Chevalley-Eilenberg differential associated to an infinitesimal A-action on E, and (2) all multibrackets $\lambda_k : \otimes^k \Gamma(\wedge^\bullet A^* \otimes E) \to \Gamma(\wedge^\bullet A^* \otimes E)[1]$ with $k \geq 2$ are $\Gamma(\wedge^\bullet A^*)$-multilinear.*

Proposition 1.1. *Let A be a Lie algebroid over a smooth manifold M and let E be a smooth vector bundle over M. Each of the following four data is equivalent to a Kapranov A-module structure on E.*

1. *A degree 1 derivation D of the graded algebra $\Gamma(\wedge^\bullet A^* \otimes \hat{S}(E^*))$, which preserves the filtration $\Gamma(\wedge A^* \otimes \hat{S}^{\geq n}(E^*))$, satisfies $D^2 = 0$, and whose restriction to $\Gamma(\wedge^\bullet A^*)$ is the Chevalley-Eilenberg differential of the Lie algebroid A. (Here, by convention, all elements of $\hat{S}(E^*)$ have degree 0.)*
2. *An infinitesimal action of A on $\hat{S}(E^*)$ by derivations which preserve the decreasing filtration $\hat{S}^{\geq n}(E^*)$.*
3. *An infinitesimal action of A on $S(E)$ by coderivations which preserve $S^{\geq 1}(E)$ and the increasing filtration $S^{\leq n}(E)$.*

1. A (p,q)-*shuffle* is a permutation σ of the set $\{1, 2, \cdots, p+q\}$ such that $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(p)$ and $\sigma(p+1) \leq \sigma(p+2) \leq \cdots \leq \sigma(p+q)$.

2. The *Koszul sign* of a permutation σ of the (homogeneous) vectors $v_1, v_2, \dots, v_n$ is determined by the relation $v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(n)} = \varepsilon(\sigma; v_1, \cdots, v_n)\, v_1 \odot v_2 \odot \cdots \odot v_n$.

(4) *An infinitesimal action of A on E together with a sequence of morphisms of vector bundles $\boldsymbol{R}_k : S^k(E) \to A^* \otimes E$ $(k \geq 2)$ whose sum*

$$\boldsymbol{R} = \sum_{k=2}^{\infty} \boldsymbol{R}_k \in \Gamma(A^* \otimes \hat{S}(E^*) \otimes E)$$

is a solution of the Maurer-Cartan equation $d_A \boldsymbol{R} + \frac{1}{2}[\boldsymbol{R}, \boldsymbol{R}] = 0$. (Here, we consider $\Gamma(\hat{S}(E^*) \otimes E)$ as the space of formal vertical vector fields on E along the zero section and derive a natural Lie bracket on the graded vector space $\Gamma(\wedge^\bullet A^* \otimes \hat{S}(E^*) \otimes E)$.)

Characterizations (1) and (4) are related by the identity $D = d_A^{\hat{S}(E^*)} + \boldsymbol{R}$, where $d_A^{\hat{S}(E^*)}$ denotes the Chevalley-Eilenberg differential associated to the infinitesimal A-action on E, and $\boldsymbol{R}$ denotes its own action on $\Gamma(\wedge^\bullet A^* \otimes \hat{S}(E^*))$ by contraction. On the other hand, for any $k \geq 2$, the k-th Kapranov multibracket λ_k is related to the k-th component $\boldsymbol{R}_k \in \Gamma(A^* \otimes S^k E^* \otimes E)$ of the Maurer-Cartan element $\boldsymbol{R}$ through the equation

$$\lambda_k(\xi_1 \otimes e_1, \cdots, \xi_k \otimes e_k) = (-1)^{|\xi_1| + \cdots + |\xi_k|} \xi_1 \wedge \cdots \wedge \xi_k \wedge \boldsymbol{R}_k(e_1, \cdots, e_k),$$

which is valid for any $e_1, \ldots, e_k \in \Gamma(E)$ and any homogeneous elements $\xi_1, \ldots, \xi_k$ of $\Gamma(\wedge^\bullet A^*)$.

The algebraic structure described in the above proposition is related to Costello's L_∞ algebras over the differential graded algebra $(\Gamma(\wedge^\bullet A^*), d_A)$ [4], and to Yu's L_∞ algebroids [9].

Two Kapranov A-modules E_1 and E_2 over M are isomorphic if there exists an isomorphism $\Phi : S(E_1) \to S(E_2)$ of bundles of coalgebras over M, which intertwines the infinitesimal A-actions.

2. Exponential map and Poincaré-Birkhoff-Witt isomorphism

Assume $\mathscr{A}$ is a Lie subgroupoid of a Lie groupoid $\mathscr{L}$ (over the same unit space), and let A and L denote the corresponding Lie algebroids. The source map $s : \mathscr{L} \to M$ factors through the quotient of the action of $\mathscr{A}$ on $\mathscr{L}$ by multiplication from the right. Therefore, it induces a surjective submersion $s : \mathscr{L}/\mathscr{A} \to M$. Note that the zero section $0 : M \to L/A$ and the unit section $1 : M \to \mathscr{L}/\mathscr{A}$ are both embeddings of M.

Proposition 2.1. *Each choice of a splitting of the short exact sequence of vector bundles $0 \to A \to L \to L/A \to 0$ and of an L-connection ∇ on L/A extending the A-action determines an exponential map,* i.e. *a fiber bundle map*

$$\exp^\nabla : L/A \to \mathscr{L}/\mathscr{A},$$

which identifies the zero section of L/A to the unit section of $\mathscr{L}/\mathscr{A}$, whose differential along the zero section of L/A is the canonical isomorphism between L/A and the tangent bundle to the s-foliation of $\mathscr{L}/\mathscr{A}$ along the unit section, and which is locally diffeomorphic around M.

Let $\mathcal{N}(L/A)$ denote the space of all functions on L/A which, together with their derivatives of all degrees in the direction of the π-fibers, vanish along the zero section. The space of π-fiberwise differential operators on L/A along the zero

section is canonically identified to the symmetric R-algebra $\Gamma(S(L/A))$. Therefore, we have the short exact sequence of R-algebras

$$0 \to \mathcal{N}(L/A) \to C^\infty(L/A) \to \operatorname{Hom}_R\big(\Gamma(S(L/A)), R\big) \to 0. \tag{1}$$

Likewise, let $\mathcal{N}(\mathscr{L}/\mathscr{A})$ denote the space of all functions on $\mathscr{L}/\mathscr{A}$ which, together with their derivatives of all degrees in the direction of the s-fibers, vanish along the unit section. The space of s-fiberwise differential operators on $\mathscr{L}/\mathscr{A}$ along the unit section is canonically identified to the quotient of the enveloping algebra $\mathcal{U}(L)$ by the left ideal generated by $\Gamma(A)$. Therefore, we have the short exact sequence of R-algebras

$$0 \to \mathcal{N}(\mathscr{L}/\mathscr{A}) \to C^\infty(\mathscr{L}/\mathscr{A}) \to \operatorname{Hom}_R\big(\tfrac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\big) \to 0. \tag{2}$$

Since the exponential (or more precisely its dual) maps $\mathcal{N}(\mathscr{L}/\mathscr{A})$ to $\mathcal{N}(L/A)$, it induces an isomorphism of R-modules from $\operatorname{Hom}_R\big(\Gamma(S(L/A)), R\big)$ to $\operatorname{Hom}_R\big(\tfrac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\big)$.

Proposition 2.2. *Each choice of a splitting of the short exact sequence of vector bundles $0 \to A \to L \to L/A \to 0$ and of an L-connection ∇ on L/A extending the A-action determines an isomorphism of filtered R-modules*

$$\mathrm{PBW} : \Gamma(S(L/A)) \to \tfrac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$$

called Poincaré-Birkhoff-Witt map.

Remark 2.1. *In case $L = A \bowtie B$ is the Lie algebroid sum of a matched pair of Lie algebroids (A, B), the L-connection ∇ on $L/A \cong B$ extending the A-action determines a B-connection on B, the coalgebras $\tfrac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ and $\mathcal{U}(B)$ are isomorphic, and the corresponding Poincaré-Birkhoff-Witt map $\mathrm{PBW} : \Gamma(S(B)) \to \mathcal{U}(B)$ is standard (see* [7] *for instance).*

Proposition 2.3. *The Poincaré-Birkhoff-Witt map associated to a splitting $j : L/A \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to L/A \to 0$ and an L-connection ∇ on L/A satisfies $\mathrm{PBW}(1) = 1$ and, for all $b \in \Gamma(L/A)$ and $n \in \mathbb{N}$, $\mathrm{PBW}(b) = j(b)$ and $\mathrm{PBW}(b^{n+1}) = j(b) \cdot \mathrm{PBW}(b^n) - \mathrm{PBW}\big(\nabla_{j(b)}(b^n)\big)$, where b^k stands for the symmetric product $b \odot b \odot \cdots \odot b$ of k copies of b.*

Remark 2.2. *Although the construction of the Poincaré-Birkhoff-Witt map outlined above presupposes that L and A are integrable real Lie algebroids,* PBW *can be defined for any real (resp. complex) Lie pair provided one works with local (resp. formal) groupoids.*

Recall that L/A is an A-module. The representation of A on L/A, an obvious generalization of the Bott connection, induces an infinitesimal action of A on the coalgebra $\Gamma(S(L/A))$ by coderivation. Besides, multiplication by elements of $\Gamma(A)$ from the left in the coalgebra $\mathcal{U}(L)$ induces an infinitesimal action of A on the coalgebra $\tfrac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ by coderivation.

The infinitesimal actions of A on L/A and $\mathscr{L}/\mathscr{A}$ induce infinitesimal actions of A by derivations on the algebras of functions $C^\infty(L/A)$ and $C^\infty(\mathscr{L}/\mathscr{A})$ and, consequently, on the algebras of infinite jets $\operatorname{Hom}_R\big(\Gamma(S(L/A)), R\big)$ and $\operatorname{Hom}_R\big(\tfrac{\mathcal{U}(L)}{\mathcal{U}(L)A}, R\big)$.

Proposition 2.4. *(1) The space* $\operatorname{Hom}_R\big(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\big)$ *of infinite s-fiberwise jets along* M *of functions on* $\mathscr{L}/\mathscr{A}$ *is an associative algebra on which the Lie algebroid* A *acts infinitesimally by derivations. (2) The dual of the exponential map* $\mathrm{PBW}^* : \operatorname{Hom}_R\big(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\big) \to \operatorname{Hom}_R\big(\Gamma(S(L/A)), R\big)$ *is an isomorphism of associative algebras, which may or may not intertwine the infinitesimal* A*-actions.*

3. $L_\infty[1]$ algebra associated to a Lie pair

Our main result is the following

Theorem 3.1. *If* (L, A) *is a Lie pair,* i.e. *a Lie algebroid* L *together with a Lie subalgebroid* A*, then* L/A *admits a Kapranov module structure, canonical up to isomorphism, over the Lie algebroid* A*, whose* $\boldsymbol{R}_2 \in \Gamma(A^* \otimes S^2(L/A)^* \otimes L/A)$ *(see Proposition 1.1) is a 1-cocycle representative of the Atiyah class of* L/A *relative to the pair* (L, A)*.*
Moreover, when $L = A \bowtie B$ *is the Lie algebroid sum of a matched pair* (A, B) *of Lie algebroids and there exists a torsion free flat* B*-connection* ∇ *on* B*, the components of the Maurer-Cartan element* $\boldsymbol{R}$ *satisfy the recursive formula* $\boldsymbol{R}_{k+1} = \partial^\nabla \boldsymbol{R}_k$*, where* ∂^∇ *denotes the covariant differential associated to the connection.*

Sketch of proof Choose a splitting of the short exact sequence of vector bundles $0 \to A \to L \to L/A \to 0$ and an L-connection ∇ on L/A extending the A-action. Identify $\Gamma(\hat{S}(L/A)^*)$ to $\operatorname{Hom}_R\big(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\big)$ via the PBW map and pull back the infinitesimal A-action of the latter to the former. According to Proposition 1.1, the resulting A-action on $\Gamma(\hat{S}(L/A)^*)$ by derivations determines a Kapranov A-module structure on L/A. Making use of Proposition 2.3, one can check directly that $\boldsymbol{R}_2$ is a 1-cocycle representative of the Atiyah class $\alpha_{L/A}$. □

As immediate consequences, we recover the following results of [3].

Corollary 3.2. *Given a Lie algebroid pair* (L, A)*, let* $\mathcal{U}(A)$ *denote the universal enveloping algebra of the Lie algebroid* A *and let* $\mathcal{A}$ *denote the category of* $\mathcal{U}(A)$*-modules. The Atiyah class of the quotient* L/A *makes* $L/A[-1]$ *into a Lie algebra object in the derived category* $D^b(\mathcal{A})$*.*

Corollary 3.3. *Let* (L, A) *be a Lie pair and let* $\mathscr{C}$ *be a bundle (of finite or infinite rank) of associative commutative algebras on which* A *acts by derivations. There exists an* $L_\infty[1]$ *algebra structure on* $\Gamma(\wedge^\bullet A^* \otimes L/A \otimes \mathscr{C})$*, canonical up to* L_∞ *isomorphism. Moreover,* $H^{\bullet-1}(A; L/A \otimes \mathscr{C})$ *is a graded Lie algebra whose Lie bracket only depends on the Atiyah class of* L/A*.*

For $\mathscr{C} = \mathbb{C}$, the Lie bracket on the cohomology $H^{\bullet-1}(A, L/A)$ happens to be trivial.

4. An example due to Kapranov

Let X be a Kähler manifold with real analytic metric. Recall that the eigenbundles $T_X^{0,1}$ and $T_X^{1,0}$ of the complex structure $J : T_X \to T_X$ ($J^2 = -\operatorname{id}$) form a matched pair of Lie algebroids [6]. Fix a point $x \in X$. The exponential map $\exp_x^{\mathrm{LC}} : T_xX \to X$ defined using the geodesics of the Levi-Civita connection ∇^{LC} originating from the point x needs not be holomorphic.

However, Calabi constructed a holomorphic exponential map $\exp_x^{\mathrm{hol}} : T_xX \to X$ as follows [2] (see also [1]). First, extend the Levi-Civita connection $\mathbb{C}$-linearly to a

$T_X\otimes\mathbb{C}$-connection $\nabla^{\mathbb{C}}$ on $T_X\otimes\mathbb{C}$. Since X is Kähler, $\nabla^{\mathrm{LC}}J=0$ and $\nabla^{\mathbb{C}}$ restricts to a $T_X\otimes\mathbb{C}$-connection on $T_X^{1,0}$. It is easy to check that the induced $T_X^{0,1}$-connection on $T_X^{1,0}$ is the canonical infinitesimal $T_X^{0,1}$-action on $T_X^{1,0}$ — a section of $T_X^{1,0}$ is $T_X^{0,1}$-horizontal iff it is holomorphic — while the induced $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ is flat and torsion free. Now let X' denote the manifold X and let X'' denote X with the opposite complex structure $-J$. The image of the diagonal embedding $X\hookrightarrow X'\times X''$ is totally real so $X'\times X''$ can be seen as a complexification of X. The restriction of $T_{X'\times X''}$ (resp. its subbundle $T_{X'}\times X''$) along the diagonal X is precisely the complexified tangent bundle $T_X\otimes\mathbb{C}$ (resp. its subbundle $T_X^{1,0}$). (See [8] for a discussion on integration of complex Lie algebroids.) The analytic continuation of the $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ in a neighborhood of the diagonal is a holomorphic $T_{X'}\times X''$-connection on the Lie algebroid $T_{X'}\times X''$, whose exponential map $\exp_x^{\mathrm{hol}}$ at a diagonal point (x,x) takes $T_xX'\times\{x\}$ (which is $(T_X^{1,0})_x$ or T_xX) into $X'\times\{x\}$ (which is X).

Consider the Lie pair $(L=T_{X'\times X''}, A=X'\times T_{X''})$, the corresponding Lie groupoids $\mathscr{L}=(X'\times X'')\times(X'\times X'')$ and $\mathscr{A}=X'\times(X''\times X'')$, and the associated quotients $L/A=T_{X'}\times X''$ and $\mathscr{L}/\mathscr{A}=(X'\times X')\times X''$. Calabi's holomorphic exponential map $\exp^{\mathrm{hol}}$ is indeed the restriction along the diagonal of the exponential map $\exp^{\nabla^{1,0}}:L/A\to\mathscr{L}/\mathscr{A}$ associated to the $T_{X'}\times X''$-connection $\nabla^{1,0}$ on the Lie algebroid $T_{X'}\times X''$ as described in Proposition 2.1.

Taking the infinite jet of $\exp^{\mathrm{hol}}$, we obtain, as in Proposition 2.2, a Poincaré-Birkhoff-Witt map $\mathrm{PBW}^{\mathrm{hol}}:\Gamma(S(T_X^{1,0}))\to\mathcal{U}(T_X^{1,0})$. Then, pulling back the infinitesimal $T_X^{0,1}$-action on $\mathcal{U}(T_X^{1,0})$ to an infinitesimal $T_X^{0,1}$-action by coderivations on $\Gamma(S(T_X^{1,0}))$, we obtain, as in Theorem 3.1, a Kapranov $T_X^{0,1}$-module structure on $T_X^{1,0}$. In this context, the tensors $R_n\in\Omega^{0,1}\big(\mathrm{Hom}(S^nT_X^{1,0},T_X^{1,0})\big)$ are the curvature $R_2\in\Omega^{1,1}\big(\mathrm{End}(T_X^{1,0})\big)$ and its higher covariant derivatives. Hence we recover the following result of Kapranov:

Theorem 4.1 ([5]). *The Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ of a Kähler manifold is an $L_\infty[1]$ algebra. For $n\geq 2$, the n-th multibracket λ_n is the composition*

$$\Omega^{0,j_1}(T_X^{1,0})\otimes\cdots\otimes\Omega^{0,j_n}(T_X^{1,0})\to\Omega^{0,j_1+\cdots+j_n}(\otimes^nT_X^{1,0})\to\Omega^{0,j_1+\cdots+j_n+1}(T_X^{1,0})$$

of the wedge product with the map induced by $R_n\in\Omega^{0,1}\big(\mathrm{Hom}(\otimes^nT_X^{1,0},T_X^{1,0})\big)$, while λ_1 is the Dolbeault operator $\overline{\partial}:\Omega^{0,j}(T_X^{1,0})\to\Omega^{0,j+1}(T_X^{1,0})$.

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Département de mathématiques, université de Lorraine
E-mail address: `camille.laurent-gengoux@univ-lorraine.fr`

Department of Mathematics, Penn State University
E-mail address: `stienon@math.psu.edu`

Department of Mathematics, Penn State University
E-mail address: `ping@math.psu.edu`